Application of Tauberian Theorem to the Exponential Decay of the Tail Probability of a Random Variable

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Abstract

We give a sufficient condition for the exponential decay of the tail probability of a non-negative random variable. We consider the Laplace-Stieltjes transform of the probability distribution function of the random variable. We present a theorem, according to which if the abscissa of convergence of the LS transform is negative finite and the real point on the axis of convergence is a pole of the LS transform, then the tail probability decays exponentially. For the proof of the theorem, we extend and apply so-called a finite form of Ikehara's complex Tauberian theorem by Graham-Vaaler.

Keywords: Tail probability of random variable; Exponential decay; Laplace transform; Complex Tauberian theorem; Graham-Vaaler's finite form

1 Introduction

The purpose of this paper is to give a sufficient condition for the exponential decay of the tail probability of a non-negative random variable. For a non-negative random variable X, $P(X > x)$ is called the *tail probability* of X. The tail probability *decays exponentially* if the limit

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) \tag{1}$$

exists and is a negative finite value.

For the random variable X, the probability distribution function of X is denoted by $F(x) = P(X \le x)$ and the Laplace-Stieltjes transform of $F(x)$ is denoted by $\varphi(s) = \int_0^\infty e^{-sx} dF(x)$. We will give a sufficient condition for the exponential decay of the tail probability $P(X > x)$ based on analytic properties of $\varphi(s)$.

In [11], we obtained a result that the exponential decay of the tail probability $P(X > x)$ is determined by the singularities of $\varphi(s)$ on its axis of convergence. In this paper, we investigate the case where $\varphi(s)$ has a pole at the real point of the axis of convergence, and reveal the relation between analytic properties of $\varphi(s)$ and the exponential decay of $P(X > x)$.

The results obtained in this paper will be applied to queueing analysis. In general, there are two main performance measures of queueing analysis, one is the number of customers Q in the

system and the other is the sojourn time W in the system. Q is a discrete random variable and W is a continuous one. It is important to evaluate the tail probabilities $P(Q > q)$ and $P(W > w)$ for designing the buffer size or link capacity in communication networks. Even in the case that the probability distribution functions $P(Q \leq q)$ or $P(W \leq w)$ cannot be calculated explicitly, their generating functions $Q(z) = \sum_{q=0}^{\infty} P(Q = q) z^q$ or $W(s) = \int_0^{\infty} e^{-sw} dP(W \leq w)$ can be obtained explicitly in many queues. Particularly, in M/G/1 queue, $Q(z)$ and $W(s)$ are given explicitly by Pollaczek-Khinchin formula [7]. So, in this paper, we assume that we have the explicit form of a generating function and then investigate the exponential decay of the tail probability based on the analytic properties of the generating function.

Such kind of researches have been studied motivated by a requirement for evaluating a packet loss probability of a light tailed traffic in the packet switched network.

An approach by the complex analysis is seen in [3]. A sufficient condition is given in [3] for the decay of the stationary probability of an M/G/1 type Markov chain with boundary modification and the result is applied to MAP/G/1 queue. Let $\pi = (\pi_n)$ denote the stationary probability of an M/G/1 type Markov chain with boundary modification, and $\pi(z) = \sum_n \pi_n z^n$ the probability generating function of π with the radius of convergence $r > 0$. In [3], they proved a theorem that if $z = r$ is a pole of order 1 and is the only singularity on the circle of convergence $|z| = r$, then there exists $K > 0$, $\tilde{r} > r$ such that $\pi_n = K r^{-n} + O(\tilde{r}^{-n})$. Our Theorem 1 below is an extension of this theorem in [3].

In [2], for the stationary queue length Q of a queueing system which satisfies some large deviations conditions, it is shown that the $P(Q > n)$ decays as $P(Q > n) \simeq \psi \exp(-\theta n)$ with a positive constants ψ and θ. In [4], it is shown that the stationary waiting time W and queue length Q decay exponentially for a broad class of queues with stationary input and service. In [13],[14], the stationary distribution of M/G/1 or G/M/1 type Markov chains are deeply studied. In [16], the tail of the waiting time in PH/PH/c queue is investigated. In [9], a sufficient condition is given for the stationary probability of a Markov chain of GI/G/1 type to be light tailed.

This paper is organized as follows. In section II, an application of our results to queueing analysis is presented. In section III, an example of a random variable is given whose tail probability does not decay exponentially. In section IV, some Tauberian theorems are introduced and the relation to our problems is stated. In section V, some lemmas are given and our main theorem is proved in the case of a pole of order 2. In section VI, the statement of lemmas and theorems are presented for a pole of arbitrary order. Finally, we summarize out results in section VII.

Throughout this paper, we use the following symbols. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\Re$ denote the set of complex numbers, real numbers, integers, and the real part of a complex number, respectively.

2 Application to Queueing Analysis

The author already had results on the exponential decay of the tail probability for a discrete random variable [10], and those for a general random variable [11]. The main theorem in [10] is as follows.

Theorem 1 [10] *Let X be a random variable taking non-negative integral values, and $f(z)$ be the probability generating function of X. The radius of convergence of $f(z)$ is denoted by r and*

$1 < r < \infty$ *is assumed. If the singularities of* $f(z)$ *on the circle of convergence* $|z| = r$ *are only a finite number of poles, then*

$$\lim_{n\to\infty} \frac{1}{n} \log P(X > n) = -\log r. \tag{2}$$

We can apply Theorem 1 to queueing analysis as follows.

Consider, for example, the number of customers Q in the steady state of M/D/1 queue with traffic intensity ρ. The probability generating function $Q(z) = \sum_{q=0}^{\infty} P(Q = q) z^q$ of Q is given by Pollaczek-Khinchin formula [7]:

$$Q(z) = \frac{(1-\rho)(z-1)\exp(\rho(z-1))}{z - \exp(\rho(z-1))}. \tag{3}$$

The radius of convergence of $Q(z)$ is equal to the unique solution $z = r > 1$ of the equation $z - \exp(\rho(z-1)) = 0$. Since $Q(z)$ is meromorphic in the whole finite complex plane $|z| < \infty$, in particular, the singularities of $Q(z)$ on the circle of convergence $|z| = r$ are only a finite number of poles. Therefore, by Theorem 1, we know that the tail probability $P(Q > q)$ decays exponentially as $q \to \infty$.

Next, in [11], the exponential decay of the tail probability $P(X > x)$ is investigated for a general non-negative real valued random variable X. The main theorem in [11] is as follows.

Theorem 2 [11] *Let* X *be a non-negative random variable, and* $\varphi(s)$ *be the Laplace-Stieltjes transform of the probability distribution function of* X*. The abscissa of convergence of* $\varphi(s)$ *is denoted by* σ_0 *and* $-\infty < \sigma_0 < 0$ *is assumed. If the singularities of* $\varphi(s)$ *on the axis of convergence* $\Re s = \sigma_0$ *are only a finite number of poles, then we have*

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) = \sigma_0. \tag{4}$$

Let us apply Theorem 2 to queueing analysis. In this case, however, the situation is somewhat different from that in the case of discrete random variable.

Consider the sojourn time W in M/D/1 queue with traffic intensity ρ. Writing $W(s)$ as the Laplace-Stieltjes transform of the probability distribution function $P(W \le w)$ of W, we have [7]

$$W(s) = \frac{(1-\rho)s\exp(-s)}{s - \rho + \rho\exp(-s)}. \tag{5}$$

The abscissa of convergence of $W(s)$ is the unique negative solution $s = \sigma_0$ of the equation $s - \rho + \rho\exp(-s) = 0$. We can see that the singularity of $W(s)$ on the axis of convergence $\Re s = \sigma_0$ is only a simple pole $s = \sigma_0$ [12]. In fact, the location of the poles of $W(s)$ are shown in Figure 1. The abscissa of convergence is $\sigma_0 = -1.26$ for $\rho = 0.5$. Though, it is not easy to prove that $s = \sigma_0$ is the only singularity of $W(s)$ on the axis of convergence. In order to prove it, we need some theorems such as Rouché's theorem.

The statement of Theorem 1 and 2 are formally quite the same, but to verify that the assumption in Theorem 2 holds is more difficult than Theorem 1. This is because of the difference of the convergence regions. In Theorem 1, the boundary of the convergence region of a power series is a

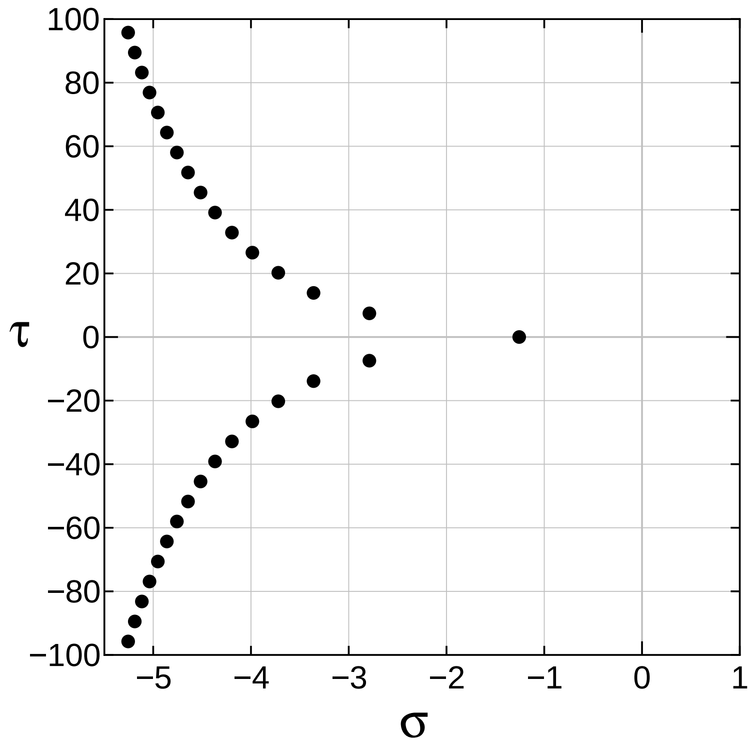

Figure 1: The poles of $W(s)$ in (5) with $\rho = 0.5,\ s = \sigma + i\tau$

circle, which is a compact set, so if the probability generating function is meromorphic, then the singularities on the circle of convergence are necessarily a finite number of poles. On the other hand, in Theorem 2, the axis of convergence is not a compact set, so we need some verification to see that the singularities on the axis of convergence are a finite number of poles. We want some simple sufficient condition to guarantee the exponential decay of the tail probability. We see, in fact, that the conclusion of Theorem 2 is really stronger than desired, so it may be possible to relax the assumption that the number of poles on the axis of convergence is finite.

We have the following theorem. This is the main theorem in this paper.

Theorem 3 *Let X be a non-negative random variable, and $F(x) = P(X \le x)$ be the probability distribution function of X. Let*

$$\varphi(s) = \int_0^\infty e^{-sx} dF(x),\ \ s = \sigma + i\tau \in \mathbb{C} \tag{6}$$

be the Laplace-Stieltjes transform of $F(x)$ and σ_0 be the abscissa of convergence of $\varphi(s)$. We assume $-\infty < \sigma_0 < 0$. If $s = \sigma_0$ is a pole of $\varphi(s)$, then we have

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) = \sigma_0. \tag{7}$$

Remark 1 *Since $F(x)$ is non-decreasing, $s = \sigma_0$ is a singularity of $\varphi(s)$ by Widder* [17], *p.58, Theorem 5b. We assume in Theorem 3 that this singularity is a pole.*

3 Example of Random Variable whose Tail Probability does not Decay Exponentially

We show an example of a non-negative random variable whose tail probability does not decay exponentially, i.e., $x^{-1} \log P(X > x)$ does not have a limit [11].

For any positive integer h, define a sequence $\{c_n\}_{n=0}^{\infty}$ by

$$\begin{cases} c_0 = 0, \\ c_n = c_{n-1} + h^{c_{n-1}}, \quad n = 1, 2, \ldots. \end{cases} \tag{8}$$

We define a function $\gamma(x)$ by

$$\gamma(x) = h^{-c_n}, \text{ for } c_n \leq x < c_{n+1}, \; n = 0, 1, \cdots. \tag{9}$$

For arbitrary $\sigma_0 < 0$, put $F^*(x) = 1 - e^{\sigma_0 x}\gamma(x), \; x \geq 0$. We see that $F^*(x)$ is right continuous and non-decreasing with $F^*(0) = 0$, $F^*(\infty) = 1$, hence $F^*(x)$ is a distribution function. Let us define X^* as a random variable with probability distribution function $F^*(x)$. We write $\varphi^*(s)$ as the Laplace-Stieltjes transform of $F^*(x)$. The following theorem shows that X^* is an example of a random variable whose tail probability does not decay exponentially.

Theorem 4 (see [11]) *Let X^*, $F^*(x)$ and $\varphi^*(s)$ be defined as above. Then, the abscissa of convergence of $\varphi^*(s)$ is σ_0, and we have*

$$\begin{aligned} \liminf_{x \to \infty} \frac{1}{x} \log P(X^* > x) &\leq \sigma_0 - \log h \\ &< \sigma_0 \\ &= \limsup_{x \to \infty} \frac{1}{x} \log P(X^* > x). \end{aligned}$$

All the points on the axis of convergence $\Re s = \sigma_0$ are singularities of $\varphi(s)$.

Remark 2 *From Widder* [17], *p.44, Theorem* 2.4*e*, $\limsup_{x \to \infty} x^{-1} \log P(X > x) = \sigma_0$ *holds under the condition $\sigma_0 < 0$ and no other condition is necessary. Meanwhile, Theorem 4 implies that some additional condition is required for* $\liminf_{x \to \infty} x^{-1} \log P(X > x) = \sigma_0$. *In our Theorem 3, the additional condition is the analytic property of $\varphi(s)$. The example X^* in Theorem* 4 *is in a sense pathological, so we can expect that the exponential decay is guaranteed by some weak condition.*

4 Tauberian Theorems of Laplace Transform

Theorem 3 deals with an issue of how the analytic properties of the Laplace-Stieltjes transform $\varphi(s)$ determines the asymptotic behavior of the tail probability $P(X > x)$. $\varphi(s)$ converges in the region $\Re s > \sigma_0$ and defines an analytic function in this region. Thus, according to Widder [17], p.40, Theorem 2.2b, we see that $P(X > x) = o(e^{\sigma x})$ as $x \to \infty$ for $\sigma_0 < \sigma < 0$. The main problem is whether $P(X > x)$ decays as $P(X > x) = O(e^{\sigma_0 x})$ as $x \to \infty$. So, it is appropriate to apply Tauberian theorems of Laplace transform to this problem.

In general, the relation between a function f and its transform Tf (such as power series, Laplace transform, etc.) is investigated by Abelian theorems or Tauberian theorems. In Abelian theorems, the asymptotic behavior of Tf is studied from the asymptotic behavior of f. Conversely, in Tauberian theorems, the asymptotic behavior of f is studied from that of Tf. In Tauberian theorems, generally, some additional condition is required for f. Such an additional condition is called a Tauberian condition. See [8] for the survey of the history and recent developments of Tauberian theory.

The following is a well-known Tauberian theorem.

Tauberian Theorem (*Widder* [17], *p.*187, *Theorem* 3b) *For a normalized function $\mu(x)$ of bounded variation in $[0, L]$ for every $L > 0$, let the integral*

$$\varphi(s) = \int_0^\infty e^{-sx} d\mu(x)$$

exist for $s > 0$ and let $\lim_{s\to 0+} \varphi(s) = A$. Then

$$\lim_{x\to\infty} \mu(x) = A$$

holds if and only if

$$\int_0^x t d\mu(t) = o(x), \ x \to \infty. \tag{10}$$

In the above theorem, Tauberian condition is (10). So, in this theorem, to study the asymptotic property of a function μ, other asymptotic property is assumed. I think that this type of Tauberian theorem is not easy to apply to some practical problems.

Ikehara first succeeded [6] in removing such asymptotic conditions from Tauberian theorem, instead, he posed some analytic properties of $\varphi(s)$ on the boundary of convergence region.

4.1 Ikehara's Tauberian Theorem

The following is Ikehara's Tauberian theorem, in which an analytic property of Laplace-Stieltjes transform is assumed. The Tauberian condition is the non-decreasing property of the function $S(t)$.

Theorem (*Ikehara* [6], *see also* [8]) *Let $S(t)$ vanish for $t < 0$, be non-decreasing, right continuous, and the integral*

$$\varphi(s) = \int_0^\infty e^{-st} dS(t), \ s = \sigma + i\tau \tag{11}$$

exist for $\sigma > 1$. There exists a constant A such that the analytic function

$$\varphi_0(s) = \varphi(s) - \frac{A}{s-1}, \ \Re s > 1 \tag{12}$$

converges as $\sigma \downarrow 1$ to the boundary function $\varphi_0(1 + i\tau)$ uniformly (or in L^1) for $-\lambda < \tau < \lambda$ with any $\lambda > 0$. Then we have

$$\lim_{t\to\infty} e^{-t} S(t) = A. \tag{13}$$

4.2 Finite Form of Ikehara's Theorem by Graham-Vaaler

In Ikehara's theorem, since $\lambda > 0$ is arbitrary, $\varphi(s)$ is assumed to be analytic on the whole axis of convergence $\Re s = \sigma_0$ except the pole $s = \sigma_0$. As mentioned previously, because the axis of convergence is not compact, it is difficult to check whether $\varphi(s)$ satisfies the theorem assumption or not. The following extension by Graham-Vaaler [5] solves this difficulty by relaxing the limit (13) of $e^{-t}S(t)$. This theorem is called a finite form of Ikehara's theorem because λ is restricted to some range of values.

We make preliminary definitions in order to state Graham-Vaaler's theorem.

For $\omega > 0$, define a function $E_\omega(t)$ by

$$E_\omega(t) = \begin{cases} e^{-\omega t}, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{14}$$

For $\lambda > 0$, a real function $f(x)$ is *of type* λ if $f(x)$ is the restriction to $\mathbb{R}$ of an entire function $f(z)$ of exponential type λ. An entire function $f(z)$ is *of exponential type* λ [15] if it satisfies

$$|f(z)| \leq C \exp(\lambda |z|), \;\; z \in \mathbb{C}, \;\; C > 0, \;\; \lambda > 0. \tag{15}$$

A function $f(x)$ is a *majorant* for a function $g(x)$ if $f(x) \geq g(x)$ for any $x \in \mathbb{R}$, and $f(x)$ is a *minorant* for $g(x)$ if $f(x) \leq g(x)$ for any $x \in \mathbb{R}$.

Theorem (*Graham-Vaaler* [5], *see also* [8]) *Let $S(t)$ vanish for $t < 0$, be non-decreasing, right continuous, and the integral*

$$\varphi(s) = \int_0^\infty e^{-st} dS(t), \;\; s = \sigma + i\tau \tag{16}$$

exist for $\sigma > 1$. There exists a constant A such that the analytic function

$$\varphi_0(s) = \varphi(s) - \frac{A}{s-1}, \;\; \Re s > 1 \tag{17}$$

converges as $\sigma \downarrow 1$ to the boundary function $\varphi_0(1 + i\tau)$ uniformly (or in L^1) for $-\lambda < \tau < \lambda$ with some $\lambda > 0$. Then, for any majorant $M(t)$ for $E_1(t)$ of type λ and any minorant $m(t)$ for $E_1(t)$ of type λ, we have

$$A \int_{-\infty}^{\infty} m(t) dt \leq \liminf_{t \to \infty} e^{-t} S(t) \tag{18}$$

$$\leq \limsup_{t \to \infty} e^{-t} S(t) \leq A \int_{-\infty}^{\infty} M(t) dt. \tag{19}$$

5 Main Theorem

We write below our main theorem again. By this theorem, we do not need to check the location of singularities of the LS transform $\varphi(s)$ and if $\varphi(s)$ is meromorphic then the assumption of the theorem is necessarily satisfied.

Theorem 3 *Let X be a non-negative random variable, and $F(x) = P(X \leq x)$ be the probability distribution function of X. Let*

$$\varphi(s) = \int_0^\infty e^{-sx} dF(x) \tag{20}$$

be the Laplace-Stieltjes transform of $F(x)$ and σ_0 be the abscissa of convergence of $\varphi(s)$. We assume $-\infty < \sigma_0 < 0$. If $s = \sigma_0$ is a pole of $\varphi(s)$, then we have

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) = \sigma_0. \tag{21}$$

It is possible to give a proof for arbitrary order of poles, but the description becomes very complicated, so we will prove only for the pole of order 2. A proof for higher order poles is easily obtained from the proof for the pole of order 2.

5.1 Preliminary Lemmas for the Proof of Main Theorem

For $\omega > 0$, we define functions

$$R(v) = \frac{v^3}{1-e^{-v}}, \;\; v \in \mathbb{R}, \tag{22}$$

$$\tilde{R}_\omega(v) = R(v+\omega) - R(\omega) - R'(\omega)v - \frac{R''(\omega)}{2}v^2, \;\; v \in \mathbb{R}. \tag{23}$$

By calculation, we have

$$R'(v) = \frac{3v^2}{1-e^{-v}} - \frac{v^3e^{-v}}{(1-e^{-v})^2}, \tag{24}$$

$$R''(v) = \frac{6v}{1-e^{-v}} - \frac{6v^2e^{-v}}{(1-e^{-v})^2} + \frac{2v^3e^{-2v}}{(1-e^{-v})^3} + \frac{v^3e^{-v}}{(1-e^{-v})^2}, \tag{25}$$

$$\begin{aligned} R'''(v) = &\frac{6}{1-e^{-v}} - \frac{18ve^{-v}}{(1-e^{-v})^2} + \frac{18v^2e^{-2v}}{(1-e^{-v})^3} + \frac{9v^2e^{-v}}{(1-e^{-v})^2} - \frac{6v^3e^{-3v}}{(1-e^{-v})^4} - \frac{6v^3e^{-2v}}{(1-e^{-v})^3} \\ &- \frac{v^3e^{-v}}{(1-e^{-v})^2}, \end{aligned} \tag{26}$$

$$\begin{aligned} R^{(4)}(v) = &\frac{-24e^{-v}}{(1-e^{-v})^2} + \frac{72ve^{-2v}}{(1-e^{-v})^3} + \frac{36ve^{-v}}{(1-e^{-v})^2} - \frac{72v^2e^{-3v}}{(1-e^{-v})^4} - \frac{72v^2e^{-2v}}{(1-e^{-v})^3} - \frac{12v^2e^{-v}}{(1-e^{-v})^2} \\ &+ \frac{24v^3e^{-4v}}{(1-e^{-v})^5} + \frac{36v^3e^{-3v}}{(1-e^{-v})^4} + \frac{14v^3e^{-2v}}{(1-e^{-v})^3} + \frac{v^3e^{-v}}{(1-e^{-v})^2}. \end{aligned} \tag{27}$$

We have the following lemmas.

Lemma 1 *There exists $\omega_0 > 0$ such that for any $\omega \ge \omega_0$,*

$$\begin{cases} \tilde{R}_\omega(v) \ge 0, & v \ge 0, \\ \tilde{R}_\omega(v) \le 0, & v < 0. \end{cases} \tag{28}$$

Proof: From (26), we have $R'''(v) \simeq 6$ for sufficiently large $v > 0$, or more precisely, for any $\epsilon > 0$ there exists $v_0 > 0$ such that $|R'''(v) - 6| < \epsilon$ holds for $v \ge v_0$. Hence, by taking ϵ sufficiently small, we have $R'''(v) > 5 > 0$ for $v \ge v_0$, especially, $R''(v)$ is monotonic increasing for $v \ge v_0$. Further, from (25) we have

$$\lim_{v\to\infty} R''(v) = \infty, \tag{29}$$

$$\lim_{v\to-\infty} R''(v) = 0. \tag{30}$$

Write $C \equiv \max_{v \le v_0} R''(v)$, which is finite by (30). From (29), there exists ω_0 with $R''(\omega_0) = C + 1$ and $\omega_0 \ge v_0$. Then, we have

$$\max_{v \le \omega_0} R''(v) = \max\left(\max_{v \le v_0} R''(v),\ \max_{v_0 \le v \le \omega_0} R''(v)\right) \tag{31}$$
$$= \max(C, C+1) \tag{32}$$
$$= C + 1 \tag{33}$$
$$= R''(\omega_0). \tag{34}$$

Since $R''(v)$ is monotonic increasing for $v \ge \omega_0 (\ge v_0)$, we have

$$\max_{v \le \omega} R''(v) = R''(\omega),\ \forall \omega \ge \omega_0. \tag{35}$$

Next, for $\omega \ge \omega_0$, we have

$$\max_{v \ge \omega} R''(v) = R''(\omega),\ \forall \omega \ge \omega_0 \tag{36}$$

because $R''(v)$ is monotonic increasing for $v \ge \omega (\ge \omega_0 \ge v_0)$. From (23), we have $\tilde{R}''_\omega(v) = R''(v + \omega) - R''(\omega)$, thus for $\omega \ge \omega_0$, we have from (35), (36),

$$\tilde{R}''_\omega(v) \begin{cases} \ge 0, & v \ge 0, \\ \le 0, & v < 0. \end{cases} \tag{37}$$

From $\tilde{R}'_\omega(0) = 0$, we have $\tilde{R}'_\omega(v) \ge 0,\ v \in \mathbb{R}$. So, $\tilde{R}_\omega(v)$ is monotonic increasing in v. Then, $\tilde{R}_\omega(0) = 0$ implies (28).

Lemma 2 *For sufficiently large $v > 0$,*

$$R'(v) < \frac{3v^2}{1 - e^{-v}}, \tag{38}$$
$$R'''(v) < \frac{6}{1 - e^{-v}}. \tag{39}$$

Proof: (38) is easily obtained from (24). We have from (26)

$$R'''(v) = \frac{6}{1 - e^{-v}} - v^3 e^{-v}\left(1 + O(v^{-1})\right),\ v \to \infty, \tag{40}$$

thus, (39) holds.

Lemma 3 *There exists $\omega_0 > 0$ such that for any $\omega \ge \omega_0$,*

$$\tilde{R}_\omega(v) \le \frac{1}{1 - e^{-\omega}} v^3,\ v \ge 0. \tag{41}$$

Proof: By the mean value theorem and Lemma 2, there exists v_0 with $\omega \le v_0 \le v + \omega$ such that

$$\frac{\tilde{R}_\omega(v)}{v^3} = \frac{R'''(v_0)}{3!}$$
$$< \frac{1}{3!} \cdot \frac{6}{1 - e^{-v_0}} \tag{42}$$
$$\le \frac{1}{1 - e^{-\omega}}. \tag{43}$$

Lemma 4 *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,*

$$-\int_{-\infty}^{0} \tilde{R}_\omega(v) e^{tv} dv < \frac{R'''(\omega)}{t^4},\ t > 0. \tag{44}$$

Proof: By calculation, for $\omega > 0$

$$\int_{-\infty}^{0} \tilde{R}_\omega(v) e^{tv} dv = \int_{-\infty}^{0} R(v+\omega) e^{tv} dv - \frac{R(\omega)}{t} + \frac{R'(\omega)}{t^2} - \frac{R''(\omega)}{t^3}, \tag{45}$$

and from integration by parts

$$\int_{-\infty}^{0} R(v+\omega) e^{tv} dv = \frac{R(\omega)}{t} - \frac{R'(\omega)}{t^2} + \frac{R''(\omega)}{t^3} - \frac{R'''(\omega)}{t^4} + \frac{1}{t^4}\int_{-\infty}^{0} R^{(4)}(v+\omega) e^{tv} dv, \tag{46}$$

where $R^{(4)}(v)$ denotes the fourth derivative of $R(v)$. Hence from (45), (46), we have

$$\begin{aligned} \frac{R'''(\omega)}{t^4} + \int_{-\infty}^{0} \tilde{R}_\omega(v) e^{tv} dv &= \frac{1}{t^4}\int_{-\infty}^{0} R^{(4)}(v+\omega) e^{tv} dv \\ &= \frac{e^{-\omega t}}{t^4}\int_{-\infty}^{\omega} R^{(4)}(v) e^{tv} dv. \end{aligned}$$

Then, we will prove that there exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,

$$\int_{-\infty}^{\omega} R^{(4)}(v) e^{tv} dv > 0,\ t \geq 0. \tag{47}$$

We divide into two cases $t > 1$ and $0 \leq t \leq 1$.

First, let us consider the case $t > 1$. From (27), we have

$$R^{(4)}(v) = \begin{cases} v^3 e^{-v}\left(1 + O(v^{-1})\right), & v \to \infty, \\ O(v^3 e^v), & v \to -\infty, \end{cases} \tag{48}$$

especially, $R^{(4)}(v)$ is bounded in $\mathbb{R}$. From (48) we see that there exist constants $C > 0$, $v_0 > 0$ with

$$R^{(4)}(v) \geq \begin{cases} \frac{1}{2} v^3 e^{-v}, & v \geq v_0, \\ -C, & v < v_0. \end{cases} \tag{49}$$

Thus, for $t > 1$,

$$\int_{-\infty}^{\omega} R^{(4)}(v) e^{tv} dv \geq -C\int_{-\infty}^{v_0} e^{tv} dv + \frac{1}{2}\int_{v_0}^{\omega} v^3 e^{(t-1)v} dv \tag{50}$$

$$\geq -Ce^{tv_0} + \frac{v_0^3}{2}\frac{e^{\omega(t-1)} - e^{v_0(t-1)}}{t-1} \tag{51}$$

$$= \frac{e^{\omega(t-1)} - e^{v_0(t-1)}}{t-1}\left\{\frac{v_0^3}{2} - \frac{Ce^{tv_0}}{e^{\omega(t-1)} - e^{v_0(t-1)}}\right\}. \tag{52}$$

For any $\epsilon > 0$, putting $\omega_0 = v_0 + e^{v_0}/\epsilon$, we have for any $\omega \geq \omega_0$,

$$e^{\omega(t-1)} \geq e^{(v_0 + e^{v_0}/\epsilon)(t-1)} \tag{53}$$

$$\geq e^{v_0(t-1)}\left\{1 + \frac{e^{v_0}}{\epsilon}(t-1)\right\},\ t > 1, \tag{54}$$

or

$$\frac{(t-1)e^{v_0 t}}{e^{\omega(t-1)}-e^{v_0(t-1)}} \le \epsilon,\ t>1. \tag{55}$$

Therefore, we have from (52),(55), by taking ϵ sufficiently small, for $\omega \ge \omega_0$,

$$\int_{-\infty}^{\omega} R^{(4)}(v)e^{tv}dv > 0,\ t>1. \tag{56}$$

Next, we consider $0 \le t \le 1$. We first restrict $0<t<1$. The cases $t=0$ and $t=1$ will be considered later. We calculate

$$\int_{-\infty}^{\infty} R^{(4)}(v)e^{tv}dv,\ 0<t<1. \tag{57}$$

Here, we put $t=-s$ and

$$\phi(s)=\int_{-\infty}^{\infty} R^{(4)}(v)e^{-sv}dv,\ -1<s<0. \tag{58}$$

$\phi(s)$ is the bilateral Laplace transform [17] of $R^{(4)}(v)$. The region of convergence is $-1<\Re s<1$.

We apply the following theorem.

Theorem A (*Widder* [17], *p.239, Theorem* 3*d*) *Let $\alpha(v)$ be of bounded variation on any finite interval. If the integral*

$$f(s)=\int_{-\infty}^{\infty} e^{-sv}d\alpha(v) \tag{59}$$

exists for $s=s_0,\ \Re s_0<0$, and $\alpha(\infty)=0$, then

$$f(s_0)=s_0\int_{-\infty}^{\infty} e^{-s_0 v}\alpha(v)dv. \tag{60}$$

(end of theorem citation)

Let us define a step function $\Delta(v)$ as

$$\Delta(v)=\begin{cases} 1, & v\ge 0,\\ 0, & v<0,\end{cases} \tag{61}$$

and define $\alpha_1(v)=R'''(v)-6\Delta(v)$. Then,

$$\int_{-\infty}^{\infty} e^{-sv}d\alpha_1(v)=\int_{-\infty}^{\infty} e^{-sv}R^{(4)}(v)dv-6<\infty, \tag{62}$$

and $\alpha_1(\infty)=0$, so $\alpha_1(v)$ satisfies the assumptions of the above Theorem A. Hence, we have from Theorem A and (62)

$$\int_{-\infty}^{\infty} e^{-sv}R^{(4)}(v)dv-6=s\int_{-\infty}^{\infty} e^{-sv}\alpha_1(v)dv. \tag{63}$$

Next, define $\alpha_2(v)=R''(v)-6v\Delta(v)$, then from (62), (63)

$$\int_{-\infty}^{\infty} e^{-sv}d\alpha_2(v)=\int_{-\infty}^{\infty} e^{-sv}\alpha_1(v)dv<\infty, \tag{64}$$

and $\alpha_2(\infty) = 0$, so $\alpha_2(v)$ satisfies the assumptions of Theorem A. Hence, we have from Theorem A and (64)

$$\int_{-\infty}^{\infty} e^{-sv}\alpha_1(v)dv = s\int_{-\infty}^{\infty} e^{-sv}\alpha_2(v)dv. \tag{65}$$

In a similar way, by defining $\alpha_3(v) = R'(v) - 3v^2\Delta(v)$, $\alpha_4(v) = R(v) - v^3\Delta(v)$, we have from Theorem A

$$\int_{-\infty}^{\infty} e^{-sv}\alpha_2(v)dv = s\int_{-\infty}^{\infty} e^{-sv}\alpha_3(v)dv, \tag{66}$$

$$\int_{-\infty}^{\infty} e^{-sv}\alpha_3(v)dv = s\int_{-\infty}^{\infty} e^{-sv}\alpha_4(v)dv. \tag{67}$$

Therefore, from (63), (65), (66), (67) we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-sv}R^{(4)}(v)dv - 6 &= s^4\int_{-\infty}^{\infty} e^{-sv}\alpha_4(v)dv \\ &= s^4\left\{\int_{-\infty}^{0}\frac{v^3e^{-sv}}{1-e^{-v}}dv + \int_0^{\infty}\frac{v^3e^{-sv}}{e^v-1}dv\right\},\ -1 < s < 0.\end{aligned}$$

By the change of variables $t = -s$,

$$\int_{-\infty}^{\infty} R^{(4)}(v)e^{tv}dv = 6 + t^4\left\{\int_{-\infty}^{0}\frac{v^3e^{tv}}{1-e^{-v}}dv + \int_0^{\infty}\frac{v^3e^{tv}}{e^v-1}dv\right\} \tag{68}$$

$$= 6 + t^4\left\{\int_0^{\infty}\frac{v^3e^{-tv}}{e^v-1}dv + \int_0^{\infty}\frac{v^3e^{tv}}{e^v-1}dv\right\} \tag{69}$$

$$> 6,\ 0 < t < 1. \tag{70}$$

Next consider $t = 0$. Define a function $g(v)$ as

$$g(v) = \begin{cases} \left|R^{(4)}(v)\right| e^{v/2}, & v \geq 0, \\ \left|R^{(4)}(v)\right|, & v < 0. \end{cases} \tag{71}$$

Then from (48), $g(v)$ is integrable and $|R^{(4)}(v)|e^{tv} \leq g(v)$, $v \in \mathbb{R}$, for $0 < t < 1/2$. By the dominating convergence theorem

$$\begin{aligned}\int_{-\infty}^{\infty} R^{(4)}(v)dv &= \lim_{t\to 0+}\int_{-\infty}^{\infty} R^{(4)}(v)e^{tv}dv \\ &= 6 + \lim_{t\to 0} t^4 \cdot \lim_{t\to 0+}\left(\int_{-\infty}^{0}\frac{v^3e^{tv}}{1-e^{-v}}dv + \int_0^{\infty}\frac{v^3e^{tv}}{e^v-1}dv\right) \\ &= 6 + \lim_{t\to 0} t^4 \cdot \left(\int_{-\infty}^{0}\frac{v^3}{1-e^{-v}}dv + \int_0^{\infty}\frac{v^3}{e^v-1}dv\right) \\ &= 6. \end{aligned} \tag{72}$$

Last, for $t = 1$, we have from (48)

$$\int_{-\infty}^{\infty} R^{(4)}(v)e^{v}dv = \infty. \tag{73}$$

Summarizing (70), (72), (73), we obtain

$$\int_{-\infty}^{\infty} R^{(4)}(v)e^{tv}dv \geq 6,\ 0 \leq t \leq 1. \tag{74}$$

Now we show that there exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,

$$\int_{-\infty}^{\omega} R^{(4)}(v) e^{tv} dv > 5, \ 0 \leq t \leq 1. \tag{75}$$

Define

$$\phi_\omega(t) = \int_{-\infty}^{\omega} R^{(4)}(v) e^{tv} dv, \ 0 \leq t \leq 1, \ \omega > 0. \tag{76}$$

For a fixed $\omega > 0$, the integral in (76) converges for $t > -1$. In fact, from (27), for any $\delta > 0$,

$$\left| R^{(4)}(v) \right| \leq e^{(1-\delta)v} \tag{77}$$

holds for any $v < 0$ with sufficiently large absolute value. By the change of variables $v = \omega - u$ in the integral in (76), we have

$$\phi_\omega(t) = e^{t\omega} \int_0^{\infty} R^{(4)}(\omega - u) e^{-tu} du. \tag{78}$$

Hence from (77), the integral in (78) converges for $t > -1$. Then from Widder [17], p.57, Theorem 5a, we see that $\phi_\omega(t)$ is analytic for $\Re t > -1$, especially $\phi_\omega(t)$ is continuous for $0 \leq t \leq 1$.

For a fixed t with $0 \leq t \leq 1$, from (48) there exists $\omega^* > 0$ (which does not depend on t) such that for $\omega > \omega^*$, $\phi_\omega(t)$ is monotonic increasing in ω.

From (74) for any t with $0 \leq t \leq 1$, there exists $\omega(t) \geq \omega^*$ such that

$$\phi_{\omega(t)}(t) = \int_{-\infty}^{\omega(t)} R^{(4)}(v) e^{tv} dv > 5. \tag{79}$$

By the continuity property of $\phi_{\omega(t)}(u)$ in u, there is $\epsilon = \epsilon(t, \omega(t)) > 0$ such that

$$\phi_{\omega(t)}(u) > 5, \ u \in (t - \epsilon, \ t + \epsilon) \cap [0, 1]. \tag{80}$$

Define $U_t = (t - \epsilon, \ t + \epsilon) \cap [0, 1]$, then U_t is an open set in $[0, 1]$ and $t \in U_t$, so $\bigcup_{0 \leq t \leq 1} U_t = [0, 1]$. Since $[0, 1]$ is compact, there exist $t_1, \cdots, t_n \in [0, 1]$ with $U_{t_1} \cup \cdots \cup U_{t_n} = [0, 1]$. Let $\omega_0 = \max_{1 \leq k \leq n} \omega(t_k)$, then for any $\omega \geq \omega_0$,

$$\phi_\omega(t) \geq \phi_{\omega(t_k)}(t), \ t \in U_{t_k} \tag{81}$$
$$> 5 \tag{82}$$

holds for $t \in [0, 1]$. From (56), (82) we have completed the proof of Lemma 4.

Now, for $\omega > 0$, define two functions $M_\omega(t)$, $m_\omega(t)$ as follows.

$$M_\omega(t) = \frac{1}{6} \left(\frac{\sin \pi t}{\pi} \right)^4 Q_\omega(t), \ -\infty < t < \infty, \tag{83}$$

$$Q_\omega(t) = \sum_{n=0}^{\infty} e^{-n\omega} \left\{ \frac{6}{(t-n)^4} - \frac{6\omega}{(t-n)^3} + \frac{3\omega^2}{(t-n)^2} - \frac{\omega^3}{t-n} \right\} + \frac{R(\omega)}{t} - \frac{R'(\omega)}{t^2} + \frac{R''(\omega)}{t^3}, \tag{84}$$

$$m_\omega(t) = M_\omega(t) - \frac{1}{1 - e^{-\omega}} \left(\frac{\sin \pi t}{\pi t} \right)^4, \ -\infty < t < \infty. \tag{85}$$

Lemma 5 *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$, $M_\omega(t)$ is a majorant for $E_\omega(t)$ of type 4π and $\int_{-\infty}^{\infty} M_\omega(t)dt < \infty$, moreover, $m_\omega(t)$ is a minorant for $E_\omega(t)$ of type 4π and $\int_{-\infty}^{\infty} m_\omega(t)dt > 0$.*

Proof: First, consider $t < 0$. We have

$$\int_0^\infty R(v+\omega)e^{tv}dv = \int_\omega^\infty R(u)e^{t(u-\omega)}du \tag{86}$$

$$= e^{-\omega t}\int_\omega^\infty u^3 e^{tu}\sum_{n=0}^\infty e^{-nu}du. \tag{87}$$

From the monotone convergence theorem and integration by parts

$$\int_0^\infty R(v+\omega)e^{tv}dv = \sum_{n=0}^\infty e^{-n\omega}\left\{\frac{6}{(t-n)^4} - \frac{6\omega}{(t-n)^3} + \frac{3\omega^2}{(t-n)^2} - \frac{\omega^3}{t-n}\right\}.$$

Then, we have from (23)

$$\int_0^\infty \tilde{R}_\omega(v)e^{tv}dv = Q_\omega(t),\ t<0. \tag{88}$$

From Lemmas 1, 3, for any $\omega \geq \omega_0$

$$0 \leq \tilde{R}_\omega(v) \leq \frac{1}{1-e^{-\omega}}v^3,\ v \geq 0, \tag{89}$$

hence from (88)

$$0 \leq Q_\omega(t) \leq \frac{6}{1-e^{-\omega}}\cdot\frac{1}{t^4},\ t<0, \tag{90}$$

and by (83) we have

$$0 \leq M_\omega(t) \leq \frac{1}{1-e^{-\omega}}\left(\frac{\sin\pi t}{\pi t}\right)^4,\ t<0. \tag{91}$$

Next, consider $t>0,\ t\notin\mathbb{Z}$. Let us calculate $-\int_{-\infty}^0 \tilde{R}_\omega(v)e^{tv}dv$. We have

$$\int_{-\infty}^0 R(v+\omega)e^{tv}dv = e^{-\omega t}\left\{\int_{-\infty}^0 + \int_0^\omega\right\}R(u)e^{tu}du. \tag{92}$$

From the monotone convergence theorem and integration by parts

$$\int_{-\infty}^0 R(u)e^{tu}du = \sum_{n=1}^\infty \frac{6}{(t+n)^4}, \tag{93}$$

and

$$\int_0^\omega R(u)e^{tu}du = \sum_{n=0}^\infty \frac{6}{(t-n)^4} + e^{\omega t}\sum_{n=0}^\infty e^{-n\omega}\left\{\frac{\omega^3}{t-n} - \frac{3\omega^2}{(t-n)^2} + \frac{6\omega}{(t-n)^3} - \frac{6}{(t-n)^4}\right\}.$$

We obtain a formula

$$\sum_{n=-\infty}^\infty \frac{1}{(t-n)^4} = \left(\frac{\pi}{\sin\pi t}\right)^4 \tag{94}$$

by the same argument as in Ahlfors [1], p.188. From (94),

$$-\int_{-\infty}^{0} \tilde{R}_\omega(v)e^{tv}dv = Q_\omega(t) - 6e^{-\omega t}\left(\frac{\pi}{\sin \pi t}\right)^4. \tag{95}$$

Hence, from Lemmas 1, 4, for $\omega \geq \omega_0$

$$0 \leq Q_\omega(t) - 6e^{-\omega t}\left(\frac{\pi}{\sin \pi t}\right)^4 \leq \frac{R'''(\omega)}{t^4}, \tag{96}$$

and from Lemma 2

$$e^{-\omega t} \leq M_\omega(t) \leq e^{-\omega t} + \frac{1}{1-e^{-\omega}}\left(\frac{\sin \pi t}{\pi t}\right)^4, \ t > 0, \ t \notin \mathbb{Z}. \tag{97}$$

By continuity, (97) holds for all $t > 0$.

For $t = 0$, $E_\omega(0) = M_\omega(0) = 1$. Therefore from (91), (97) we see that $M_\omega(t)$ is a majorant for $E_\omega(t)$ for $\omega \geq \omega_0$.

$M_\omega(t)$ is of type 4π by the following reason. Consider a rectangle Γ_n in the complex plane with vertices $(n+1/2)+i$, $(-n-1/2)+i$, $(-n-1/2)-i$, $(n+1/2)-i$, $n = 1, 2, \cdots$. From $|\sin \pi z|^2 = \sin^2 \pi t + \sinh^2 \pi y$, $z = t + iy$, we see that $|\sin \pi z|^2$ is bounded by a constant on Γ_n, and $|Q_\omega(z)|$ is also bounded by a constant on Γ_n. Both constants are independent on n. Thus, by letting $n \to \infty$, from the maximum principle, we have for some constant C

$$|M_\omega(z)| \leq C, \ \text{for } |y| \leq 1. \tag{98}$$

Next, since $|Q_\omega(z)|$ is bounded in $|y| \geq 1$, we have

$$|M_\omega(z)| \leq C|\sin \pi z|^4, \ \text{for } |y| \geq 1. \tag{99}$$

By the formula $\sin \pi z = (e^{i\pi z} - e^{-i\pi z})/2i$, we have

$$|\sin \pi z|^4 \leq e^{4\pi|z|}, \ z \in \mathbb{C}. \tag{100}$$

From (98),(99),(100), we have

$$|M_\omega(z)| \leq Ce^{4\pi|z|}, \ z \in \mathbb{C}, \tag{101}$$

which implies $M_\omega(t)$ is of type 4π.

$\int_{-\infty}^{\infty} M_\omega(t)dt < \infty$ is trivial from (91), (97), but we calculate the integral. Note that

$$\int_{-\infty}^{\infty}\left\{\frac{R(\omega)}{t} - \sum_{n=0}^{\infty} e^{-n\omega}\frac{\omega^3}{t-n}\right\}\left(\frac{\sin \pi t}{\pi}\right)^4 dt = -\omega^3\int_{-\infty}^{\infty}\sum_{n=0}^{\infty} ne^{-n\omega}\frac{1}{t(t-n)}\left(\frac{\sin \pi t}{\pi}\right)^4 dt. \tag{102}$$

Is is easy to check the following inequalities;

$$|t(t-n)| \geq \begin{cases} (t-n)^2, & t \geq n, \\ t^2, & t < 0, \end{cases}$$

and

$$\begin{aligned} \frac{1}{|t(t-n)|}\left(\frac{\sin \pi t}{\pi}\right)^4 &= \frac{\sin^2 \pi t}{\pi^2}\left|\frac{\sin \pi t}{\pi t}\right|\left|\frac{\sin \pi(t-n)}{\pi(t-n)}\right| \\ &\leq \frac{1}{\pi^2}, \ t \in \mathbb{R}. \end{aligned}$$

Hence, we have

$$\int_{-\infty}^{\infty} \left|\frac{1}{t(t-n)}\right| \left(\frac{\sin \pi t}{\pi}\right)^4 dt \le \int_{-\infty}^{0} \frac{1}{t^2} \left(\frac{\sin \pi t}{\pi}\right)^4 dt + \int_0^n \frac{1}{\pi^2} dt + \int_n^{\infty} \frac{1}{(t-n)^2} \left(\frac{\sin \pi t}{\pi}\right)^4 dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{t^2} \left(\frac{\sin \pi t}{\pi}\right)^4 dt + \frac{n}{\pi^2},$$

From the dominating convergence theorem, we can calculate (102) as follows (by considering Cauchy's principal value)

$$\int_{-\infty}^{\infty} \left\{\frac{R(\omega)}{t} - \sum_{n=0}^{\infty} e^{-n\omega} \frac{\omega^3}{t-n}\right\} \left(\frac{\sin \pi t}{\pi}\right)^4 dt$$
$$= \omega^3 \sum_{n=0}^{\infty} e^{-n\omega} \left\{\int_{-\infty}^{\infty} \frac{1}{t} \left(\frac{\sin \pi t}{\pi}\right)^4 dt - \int_{-\infty}^{\infty} \frac{1}{t-n} \left(\frac{\sin \pi t}{\pi}\right)^4 dt\right\} \tag{103}$$
$$= 0. \tag{104}$$

Therefore, we have

$$\int_{-\infty}^{\infty} M_\omega(t) dt = \frac{1}{6} \sum_{n=0}^{\infty} e^{-n\omega} \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi}\right)^4 \frac{6}{(t-n)^4} dt + \frac{1}{6} \sum_{n=0}^{\infty} e^{-n\omega} \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi}\right)^4 \frac{3\omega^2}{(t-n)^2} dt$$
$$- \frac{1}{6} \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi}\right)^4 \frac{R'(\omega)}{t^2} dt \tag{105}$$
$$= \frac{1}{1-e^{-\omega}} \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right)^4 dt + \frac{1}{6\pi^2} \left\{\frac{3\omega^2}{1-e^{-\omega}} - R'(\omega)\right\} \int_{-\infty}^{\infty} \frac{\sin^4 \pi t}{(\pi t)^2} dt \tag{106}$$
$$< \infty. \tag{107}$$

Next, we prove for $m_\omega(t)$. From (85), (97), for $\omega \ge \omega_0$

$$e^{-\omega t} - \frac{1}{1-e^{-\omega}} \left(\frac{\sin \pi t}{\pi t}\right)^4 \le m_\omega(t) \le e^{-\omega t},\ t \ge 0 \tag{108}$$

and from (85), (91),

$$-\frac{1}{1-e^{-\omega}} \left(\frac{\sin \pi t}{\pi t}\right)^4 \le m_\omega(t) \le 0,\ t < 0 \tag{109}$$

so, $m_\omega(t)$ is a minorant for $E_\omega(t)$ and is of type 4π. Since

$$\int_{-\infty}^{\infty} m_\omega(t) dt = \int_{-\infty}^{\infty} M_\omega(t) dt - \frac{1}{1-e^{-\omega}} \int_{-\infty}^{\infty} \left(\frac{\sin \pi t}{\pi t}\right)^4 dt$$
$$= \frac{1}{6\pi^2} \left\{\frac{3\omega^2}{1-e^{-\omega}} - R'(\omega)\right\} \int_{-\infty}^{\infty} \frac{\sin^4 \pi t}{(\pi t)^2} dt,$$

from Lemma 2 for $\omega \ge \omega_0$

$$\int_{-\infty}^{\infty} m_\omega(t) dt > 0. \tag{110}$$

5.2 Extension of Graham-Vaaler's Theorem

Substituting $t \to -\frac{\sigma_0}{\omega}t$ into $E_\omega(t)$ defined by (14), we have

$$E_\omega(-\frac{\sigma_0}{\omega}t) = E_{\sigma_0}(t). \tag{111}$$

Define $\lambda = 2\pi/\omega$ and

$$M_{\lambda,\sigma_0}(t) = M_\omega(-\frac{\sigma_0}{\omega}t),\ t \in \mathbb{R},\ \omega > 0, \tag{112}$$

$$m_{\lambda,\sigma_0}(t) = m_\omega(-\frac{\sigma_0}{\omega}t),\ t \in \mathbb{R},\ \omega > 0, \tag{113}$$

then both $M_{\lambda,\sigma_0}(t)$ and $m_{\lambda,\sigma_0}(t)$ are of type $-2\sigma_0\lambda$. From Lemma 5, for sufficiently small $\lambda > 0$, $M_{\lambda,\sigma_0}(t)$ and $m_{\lambda,\sigma_0}(t)$ are majorant and minorant for $E_{\sigma_0}(t)$, respectively.

Based on Lemma 5, we can calculate

$$\int_{-\infty}^{\infty} M_{\lambda,\sigma_0}(t)dt = -\frac{\omega}{\pi\sigma_0(1-e^{-\omega})}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt - \frac{\omega}{6\sigma_0\pi^3}\left\{\frac{3\omega^2}{1-e^{-\omega}} - R'(\omega)\right\}\int_{-\infty}^{\infty}\frac{\sin^4 t}{t^2}dt \tag{114}$$

$$< \infty, \tag{115}$$

$$\int_{-\infty}^{\infty} m_{\lambda,\sigma_0}(t)dt = -\frac{\omega}{6\sigma_0\pi^3}\left\{\frac{3\omega^2}{1-e^{-\omega}} - R'(\omega)\right\}\int_{-\infty}^{\infty}\frac{\sin^4 t}{t^2}dt \tag{116}$$

$$> 0. \tag{117}$$

We have the following theorem. The proof is an extension of the proof of Graham-Vaaler's Tauberian theorem to the case that the pole is of order 2.

Theorem 5 *Let X be a non-negative random variable and $\varphi(s)$ be the Laplace-Stieltjes transform of the probability distribution function $F(x)$ of X. The abscissa of convergence of $\varphi(s)$ is denoted by σ_0 and $-\infty < \sigma_0 < 0$ is assumed. Let $s = \sigma_0$ be a pole of $\varphi(s)$ of order 2 and $\varphi(s)$ be analytic on the interval $\{s = \sigma_0 + i\tau \,|\, 2\sigma_0\lambda < \tau < -2\sigma_0\lambda\}$ for some $\lambda > 0$ except $s = \sigma_0$. Write A_2 as the coefficient of $(s-\sigma_0)^{-2}$ in the Laurent expansion of $\varphi(s)$ at $s = \sigma_0$. Then, we have*

$$\begin{aligned} A_2\int_{-\infty}^{\infty} m_{\lambda,\sigma_0}(t)dt &\le \liminf_{x\to\infty} x^{-1}e^{-\sigma_0 x}P(X > x) \\ &\le \limsup_{x\to\infty} x^{-1}e^{-\sigma_0 x}P(X > x) \\ &\le A_2\int_{-\infty}^{\infty} M_{\lambda,\sigma_0}(t)dt. \end{aligned}$$

Proof: For the evaluation of $P(X > x) = \int_x^\infty dF(t)$, we first evaluate the following integral. For $\sigma > \sigma_0$, we have

$$\begin{aligned} e^{-\sigma_0 x}\int_x^\infty e^{-(\sigma-\sigma_0)t}dF(t) &= \int_x^\infty e^{\sigma_0(t-x)}e^{-\sigma t}dF(t) \\ &= \int_0^\infty E_{\sigma_0}(t-x)e^{-\sigma t}dF(t) \\ &\le \int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t), \end{aligned} \tag{118}$$

if $F(t)$ is continuous at x. If x is a point of discontinuity of $F(t)$, then the integral in (118) does not exist. Since $F(t)$ is right continuous, the set of points of discontinuity is at most countable. We see that the integral $\int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t)$ is continuous with respect to $x \in \mathbb{R}$ because $M_{\lambda,\sigma_0}(t)$ is bounded. $\int_x^\infty e^{-(\sigma-\sigma_0)t}dF(t)$ is monotonic decreasing in x. That is, a decreasing function $\int_x^\infty e^{-(\sigma-\sigma_0)t}dF(t)$ is bounded above by a continuous function $e^{\sigma_0 x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t)$ outside a countable set of x, hence, so is for all $x \in \mathbb{R}$. Therefore,

$$e^{-\sigma_0 x}\int_x^\infty e^{-(\sigma-\sigma_0)t}dF(t) \le \int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t) \tag{119}$$

holds for all $x \in \mathbb{R}$.

Since M_{λ,σ_0} is of type $-2\sigma_0\lambda$, Paley-Wiener's theorem [15] shows that the support of its Fourier transform $\hat{M}_{\lambda,\sigma_0}$ is contained in the interval $[2\sigma_0\lambda, -2\sigma_0\lambda]$. Writing $\Lambda = -2\sigma_0\lambda$, we have by the inverse Fourier transform

$$M_{\lambda,\sigma_0}(t) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(\tau)e^{it\tau}d\tau, \tag{120}$$

and by substituting $t \to t-x$, $\tau \to -\tau$,

$$M_{\lambda,\sigma_0}(t-x) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)e^{-i(t-x)\tau}d\tau. \tag{121}$$

Then we have by Fubini's theorem

$$\frac{1}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t) = \frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)e^{ix\tau}d\tau\int_0^\infty e^{-(\sigma+i\tau)t}dF(t) \tag{122}$$

$$= \frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)\varphi(\sigma+i\tau)e^{ix\tau}d\tau. \tag{123}$$

Since $s=\sigma_0$ is supposed to be a pole of order 2, the principal part $\varphi_0(s)$ of $\varphi(s)$ at $s=\sigma_0$ is given as

$$\varphi_0(s) = \sum_{j=1}^{2}\frac{A_j}{(s-\sigma_0)^j},\ A_2 \ne 0. \tag{124}$$

The inverse Laplace transform of $\varphi_0(s)$ is

$$\sum_{j=1}^{2}A_j t^{j-1}e^{\sigma_0 t},\ t \ge 0, \tag{125}$$

then, in a similar way as (123), we have

$$\frac{1}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}\sum_{j=1}^{2}A_j t^{j-1}e^{\sigma_0 t}dt = \frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)\sum_{j=1}^{2}\frac{A_j}{(\sigma+i\tau-\sigma_0)^j}e^{ix\tau}d\tau. \tag{126}$$

We write $\xi(s) = \varphi(s) - \varphi_0(s)$, then $\xi(s)$ is analytic in a neighborhood of $s=\sigma_0$. By subtracting (126) from (123),

$$\frac{1}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)e^{-\sigma t}dF(t) = \sum_{j=1}^{2}\frac{A_j}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)t^{j-1}e^{-(\sigma-\sigma_0)t}dt$$

$$+\frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)\xi(\sigma+i\tau)e^{ix\tau}d\tau. \tag{127}$$

Then from (119), (127) we have

$$\frac{1}{x}e^{-\sigma_0 x}\int_x^\infty e^{-(\sigma-\sigma_0)t}dF(t) \le \sum_{j=1}^{2}\frac{A_j}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)t^{j-1}e^{-(\sigma-\sigma_0)t}dt$$
$$+\frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)\xi(\sigma+i\tau)e^{ix\tau}d\tau. \quad (128)$$

By taking $\omega > 0$ sufficiently large, or equivalently, taking $\lambda > 0$ sufficiently small, $\xi(\sigma+i\tau) \to \xi(\sigma_0+i\tau)$ as $\sigma \downarrow \sigma_0$ uniformly in $\tau \in [-\Lambda, \Lambda]$ with $\Lambda = -2\sigma_0\lambda$. Hence by the dominating convergence theorem, we have

$$\frac{1}{x}e^{-\sigma_0 x}P(X>x) \le \sum_{j=1}^{2}\frac{A_j}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)t^{j-1}dt + \frac{1}{2\pi x}\int_{-\Lambda}^{\Lambda}\hat{M}_{\lambda,\sigma_0}(-\tau)\xi(\sigma_0+i\tau)e^{ix\tau}d\tau. \quad (129)$$

When x tends to infinity, the first term of the right hand side of (129) becomes

$$\lim_{x\to\infty}\sum_{j=1}^{2}\frac{A_j}{x}\int_0^\infty M_{\lambda,\sigma_0}(t-x)t^{j-1}dt = A_2\int_{-\infty}^{\infty}M_{\lambda,\sigma_0}(t)dt, \quad (130)$$

while the second term tends to 0 by the Riemann-Lebesgue theorem. Hence from (129) we have

$$\limsup_{x\to\infty}\frac{1}{x}e^{-\sigma_0 x}P(X>x) \le A_2\int_{-\infty}^{\infty}M_{\lambda,\sigma_0}(t)dt < \infty.$$

In a similar way, we have for minorant $m_\omega(t)$

$$\liminf_{x\to\infty}\frac{1}{x}e^{-\sigma_0 x}P(X>x) \ge A_2\int_{-\infty}^{\infty}m_{\lambda,\sigma_0}(t)dt > 0.$$

Proof of Theorem 3: From Theorem 5, writing $C_1 = A_2\int_{-\infty}^{\infty}m_{\lambda,\sigma_0}(t)dt > 0$, $C_2 = A_2\int_{-\infty}^{\infty}M_{\lambda,\sigma_0}(t)dt > 0$, we see that for any $\epsilon > 0$ there exists x_0 with

$$C_1-\epsilon < x^{-1}e^{-\sigma_0 x}P(X>x) < C_2+\epsilon,\ \forall x \ge x_0. \quad (131)$$

By taking logarithm of each side of (131), we have

$$\lim_{x\to\infty}\frac{1}{x}\log P(X>x) = \sigma_0. \quad (132)$$

We immediately obtain the following corollary of Theorem 3.

Corollary 1 *Let X be a random variable taking non-negative integral values, and $f(z) = \sum_{n=0}^{\infty}P(X=n)z^n$ the probability generating function of X. The radius of convergence of $f(z)$ is denoted by r and $1 < r < \infty$ is assumed. If $z = r$ is a pole of $f(z)$, then the tail probability $P(X>n)$ decays exponentially.*

6 Lemmas and Theorems for a Pole of Arbitrary Order

In the last section, we proved lemmas and theorems for a pole of order 2. We here provide the statement of lemmas and theorems for a pole of arbitrary order, corresponding to each lemma and theorem in the last section.

Let D be the order of the pole $s = \sigma_0$ of the Laplace-Stieltjes transform $\varphi(s)$. First, define

$$K = \begin{cases} D, & \text{if } D \text{ is odd}, \\ D+1, & \text{if } D \text{ is even}, \end{cases} \tag{133}$$

so, K is always odd. Then, define

$$R(v) = \frac{v^K}{1-e^{-v}}, \ v \in \mathbb{R}, \tag{134}$$

$$r(v) = v^K, \ v \in \mathbb{R}, \tag{135}$$

$$\tilde{R}_\omega(v) = R(v+\omega) - \sum_{k=0}^{K-1} \frac{R^{(k)}(\omega)}{k!} v^k, \ \omega > 0, \ v \in \mathbb{R}, \tag{136}$$

where $R^{(k)}$ denotes the k-th derivative of R.

We show below Lemmas 1*-5* and Theorem 5* corresponding to Lemmas 1-5 and Theorem 5 in the last section, respectively.

Lemma 1* *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,*

$$\begin{cases} \tilde{R}_\omega(v) \geq 0, & v \geq 0, \\ \tilde{R}_\omega(v) \leq 0, & v < 0. \end{cases} \tag{137}$$

Lemma 2* *For sufficiently large $v > 0$,*

$$R^{(k)}(v) < \frac{r^{(k)}(v)}{1-e^{-v}}, \ k = 1, 3, \cdots, K. \tag{138}$$

Lemma 3* *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,*

$$\tilde{R}_\omega(v) \leq \frac{1}{1-e^{-\omega}} v^K, \ v \geq 0. \tag{139}$$

Lemma 4* *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$,*

$$-\int_{-\infty}^{0} \tilde{R}_\omega(v) e^{tv} dv < \frac{R^{(K)}(\omega)}{t^{K+1}}, \ t > 0. \tag{140}$$

Define two functions $M_\omega(t)$ and $m_\omega(t)$ as follows.

$$M_\omega(t) = \frac{1}{K!} \left(\frac{\sin \pi t}{\pi} \right)^{K+1} Q_\omega(t), \ -\infty < t < \infty, \tag{141}$$

$$Q_\omega(t) = \sum_{n=0}^{\infty} e^{-n\omega} \sum_{k=1}^{K+1} \frac{(-1)^k r^{(k-1)}(\omega)}{(t-n)^k} + \sum_{k=1}^{K} \frac{(-1)^{k-1} R^{(k-1)}(\omega)}{t^k}, \tag{142}$$

$$m_\omega(t) = M_\omega(t) - \frac{1}{1-e^{-\omega}} \left(\frac{\sin \pi t}{\pi t} \right)^{K+1}, \ -\infty < t < \infty. \tag{143}$$

Lemma 5* *There exists $\omega_0 > 0$ such that for any $\omega \geq \omega_0$, $M_\omega(t)$ is a majorant for $E_\omega(t)$ of type $(K+1)\pi$ and $\int_{-\infty}^{\infty} M_\omega(t)dt < \infty$, moreover, $m_\omega(t)$ is a minorant for $E_\omega(t)$ of type $(K+1)\pi$ and $\int_{-\infty}^{\infty} m_\omega(t)dt > 0$.*

Let $\lambda = 2\pi/\omega$ and define

$$M_{\lambda,\sigma_0}(t) = M_\omega(-\frac{\sigma_0}{\omega}t), \ t \in \mathbb{R}, \tag{144}$$

$$m_{\lambda,\sigma_0}(t) = m_\omega(-\frac{\sigma_0}{\omega}t), \ t \in \mathbb{R}. \tag{145}$$

Both $M_{\lambda,\sigma_0}(t)$ and $m_{\lambda,\sigma_0}(t)$ are of type $-2\sigma_0\lambda$. By Lemma 5*, there exists $\lambda_0 > 0$ such that for any $\lambda < \lambda_0$, $M_{\lambda,\sigma_0}(t)$ is a majorant and $m_{\lambda,\sigma_0}(t)$ is a minorant for $E_{\sigma_0}(t)$, respectively.

We can calculate $\int_{-\infty}^{\infty} M_{\lambda,\sigma_0}(t)dt$ and $\int_{-\infty}^{\infty} m_{\lambda,\sigma_0}(t)dt$ as follows.

$$\begin{aligned}
\int_{-\infty}^{\infty} M_{\lambda,\sigma_0}(t)dt = &-\frac{\omega}{\pi\sigma_0(1-e^{-\omega})}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^{K+1} dt \\
&-\frac{\omega}{K!\sigma_0}\sum_{k=2,\,k:\text{even}}^{K-1}\frac{1}{\pi^{K+2-k}}\left\{\frac{r^{(k-1)}(\omega)}{1-e^{-\omega}} - R^{(k-1)}(\omega)\right\}\int_{-\infty}^{\infty}\frac{\sin^{K+1} t}{t^k}dt, && (146)\\
< \infty, &&& (147)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} m_{\lambda,\sigma_0}(t)dt = &-\frac{\omega}{K!\sigma_0}\sum_{k=2,\,k:\text{even}}^{K-1}\frac{1}{\pi^{K+2-k}}\left\{\frac{r^{(k-1)}(\omega)}{1-e^{-\omega}} - R^{(k-1)}(\omega)\right\}\int_{-\infty}^{\infty}\frac{\sin^{K+1} t}{t^k}dt && (148)\\
> 0. &&& (149)
\end{aligned}$$

Theorem 5* *Let X be a non-negative random variable and $\varphi(s)$ be the Laplace-Stieltjes transform of the probability distribution function $F(x)$ of X. The abscissa of convergence of $\varphi(s)$ is denoted by σ_0 and $-\infty < \sigma_0 < 0$ is assumed. Let $s = \sigma_0$ be a pole of $\varphi(s)$ of order D and $\varphi(s)$ be analytic on the interval $\{s = \sigma_0 + i\tau \,|\, 2\sigma_0\lambda < \tau < -2\sigma_0\lambda\}$ for some $\lambda > 0$ except $s = \sigma_0$. Write A_D as the coefficient of $(s-\sigma_0)^{-D}$ in the Laurent expansion of $\varphi(s)$ at $s = \sigma_0$. Then, we have*

$$\begin{aligned}
A_D\int_{-\infty}^{\infty} m_{\lambda,\sigma_0}(t)dt &\leq \liminf_{x\to\infty} x^{-D+1}e^{-\sigma_0 x}P(X > x) \\
&\leq \limsup_{x\to\infty} x^{-D+1}e^{-\sigma_0 x}P(X > x) \\
&\leq A_D\int_{-\infty}^{\infty} M_{\lambda,\sigma_0}(t)dt.
\end{aligned}$$

7 Conclusion

We investigated a sufficient condition for the exponential decay of the tail probability $P(X > x)$ of a non-negative random variable X. For the Laplace-Stieltjes transform $\varphi(s)$ of the probability distribution function of X with abscissa of convergence σ_0, $-\infty < \sigma_0 < 0$, we showed that if $s = \sigma_0$ is a pole of $\varphi(s)$ then the tail probability decays exponentially. If $\varphi(s)$ is given explicitly, then this sufficient condition is easy to check. For the proof of our main theorem, we extended Graham-Vaaler's Tauberian theorem to the case that the order of the pole of $\varphi(s)$ is arbitrary. Now, I have proved the conjecture that was written in [11].

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